Proceedings of the 15th International Conference on Manufacturing Systems – ICMaS
Published by Editura Academiei Române, ISSN 1842-3183

"Politehnica" University of Bucharest, Machine and Manufacturing Systems Department
Bucharest, Romania, 26–27 October, 2006

CONTRIBUTIONS AT THE THEORY OF THE GENERATION OF THE DIRECTRIX AND GENERATRIX CURVES OF THE GEOMETRICAL SURFACES

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Abstract: *In the paper is presented an original theoretical study of the cinematic generation of the complex geometrical surfaces using the composition of the single movements. It is also presented, like an application, the cinematic generation of the involutes with a fixed strait line and the cinematic generation of the conic helix, used as theoretical curves G, respectively D of the geometrical surfaces having a large utilization in technical practice, involutes surfaces and, respectively, conic helical surfaces.*

Key words: *cinematic generation, involutes, conic helix, directrix, generatrix.*

1. INTRODUCTION

The generation of the real surface on the machine tools is based on the cinematic principle of the generation of the geometrical surfaces according to their form, conform that they are generated by a generatrix curve *G*, by it movement along the directrix curve *D*.

The curves *G* and *D* with a complex geometrical form other then circle or right line (epicycloids, involutes, circular helix) can be generated cinematically using the compositions of single movements, cinematically co-coordinated in order to obtain their theoretical form.

In the paper are presented original aspects of the cinematic generation of two from the most employed curves *G* and *D* at the real generation of the surfaces on the machine tools.

2. COMPOSITIONS OF SIMPLE MOVEMENTS

The simplest curves *G* and *D* are obtained as circular or linear trajectories by the rotation or right translation of the mobile machine parts (spindle or slide) that are named simple movements.

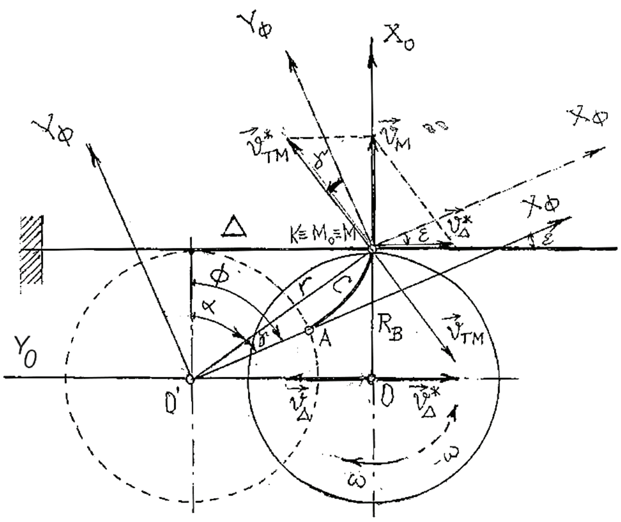

Fig. 1. Generation of involutes with a fixed right line.

For the realization of the *G* and *D* curves with complex forms, different of curves obtained with simple movements (circular or linear) in the cinematic process of generation of the real surfaces on machine tools it is necessary to employ the composition of simple movements, generating in this way any geometrical form of real surfaces. In that generation cases, between the speed parameters of the simple movements that are compounded, must exist permanently a ratio called "ratio of cinematic coordination (R_{CCIN}).

This ratio must be kept the same during all the time of the composition of respective single movements that in all the time of the composition of the real surface generation with a given geometrical form.

2.1. The composition of a rotation movement with a rectilinear translation movement made in the rotation, on a tangential direction at a circle of the rotation plane

We consider a circular base with the radius R_B and with the center *O* in the origin of the initial reference system X_0OY_0 solidary with the base (Fig. 1).

A circular rolling curve having an infinite radius (the right line Δ) is considered fixed and tangent in the point *K* at the base. The *K* point is the initial generating point M_0 and becomes the generating point *M* of the cinematic curve *C* ($M = M_0 = K$).

It is imparted to the base circle a rotation movement with a angular speed ω, and, in the some time, a rectiliniar uniform movement to it center *O* with the speed $\vec{v}_\Delta$, in a direction parallel with the right line Δ. These movements are cinematically coordinated so that the base circle rolls without sliding on the right line Δ. The cinematic co-ordination is made by R_{CCIN} that impose to the speeds the relationship:

$$R_{CCIN} = \frac{v_\Delta}{\omega} \equiv R_B \,. \tag{1}$$

For the rolling of the base circle on the rolling angle Φ, this reference system come up in the position $X_\Phi OY_\Phi$

and, in this time the generating point M generates the cinematic curve C by this resulting movement, movement relative to the base circle. The relative movement is obtained considering the base fixed by the rotation of the entire system with the angular speed ω, simultaneously with the right linear movement of the center O with the speed $\vec{v}_{\Delta}^{*}$.

By consequence, the speed of the generating point M in it relative movement, during the rolling of the base on the right line Δ is the vector $\vec{v}_M$ given by the relationship:

$$\vec{v}_M = \vec{v}_{\Delta}^{*} + \vec{v}_{TM}, \tag{2}$$

in which: $\vec{v}_{TM}$ is the vector of the tangential speed from the rotation movement with the angular speed ω, those size is:

$$\left|\vec{v}_{TM}^{*}\right| = v_{TM} = r\omega, \tag{3}$$

and the vector $\vec{v}_{\Delta}^{*}$ has the form:

$$\left|\vec{v}_{\Delta}^{*}\right| = \vec{v}_{\Delta}. \tag{4}$$

If it is projected the vectorial relation (2) on the axis co-ordinates becomes in the position $X_{\Phi}OY_{\Phi}$, we obtain the size of the speed's components $\vec{v}_M$ on the axes with the relations:

$$\begin{aligned} v_{MX} &= \left|v_{\Delta}^{*}\right|\cos\varepsilon - \left|\vec{v}_{TM}^{*}\right|\sin\gamma, \\ v_{MY} &= -\left|v_{\Delta}^{*}\right|\sin\varepsilon + \left|\vec{v}_{TM}^{*}\right|\cos\gamma. \end{aligned} \tag{5}$$

In Fig. 1 we can observe that:

$$\varepsilon = \frac{\pi}{2} - \Phi; \quad \gamma = \Phi - \alpha; \quad r = \frac{R_B}{\cos\alpha}. \tag{6}$$

If we replace the relationships (1), (3), (4) and (6) in (5), we obtain the final expressions of the sizes of the components of the speeds $\vec{v}_M$ on the co-ordinate axes of the base:

$$\begin{aligned} v_{MX} &= \omega R_B \Phi\cos\Phi, \\ v_{MY} &= \omega R_B \Phi\sin\Phi. \end{aligned} \tag{7}$$

The differential co-ordinates dX and dY of the generating point M are differential spaces of the covered distance respectively with the speeds v_{MX} and v_{MY}, in a differential time dT given by the relation:

$$\mathrm{d}T = \mathrm{d}\Phi/\mathrm{d}\omega. \tag{8}$$

So, these co-ordinates has the expressions:

$$\begin{aligned} \mathrm{d}X &= v_{MX}\mathrm{d}T, \\ dY &= v_{MY}\mathrm{d}T. \end{aligned} \tag{9}$$

We replace now the relations (7) and (8) in (9) ant we obtain the expressions of the momentary X and Y of the generating point M:

$$\begin{aligned} X &= \int v_{MX}\mathrm{d}T = \frac{1}{\omega}\int \omega R_B\Phi\cos\Phi\mathrm{d}\Phi = R_B\int \Phi\cos\Phi\mathrm{d}\Phi, \\ Y &= \int v_{MY}\mathrm{d}T = \frac{1}{\omega}\int \omega R_B\Phi\sin\Phi\mathrm{d}\Phi = R_B\int \Phi\sin\Phi\mathrm{d}\Phi. \end{aligned} \tag{10}$$

By performing the integral we obtain the co-ordinates X and Y of the point M that represent the parametrical equations of the cinematic curve C, described by the generating point M as consequence of the two movements:

$$\begin{aligned} X &= R_B\left(\cos\Phi + \Phi\sin\Phi\right) + C_X, \\ Y &= R_B\left(\sin\Phi - \Phi\cos\Phi\right) + C_Y. \end{aligned} \tag{11}$$

in which integration constants C_X and C_Y can be determinate knowing that at $\Phi = 0$, $X_0 = R_B$ and $Y_0 = 0$, resulting $C_X = 0$ and $C_Y = 0$.

By consequence, the parametric equations of the cinematic curve C are:

$$\begin{aligned} X &= R_B\left(\cos\Phi + \Phi\sin\Phi\right), \\ Y &= R_B\left(\sin\Phi - \Phi\cos\Phi\right). \end{aligned} \tag{12}$$

those represent the parametric equations of an involute.

This kind of generation of the cinematic curve C is named the generation of the involutes with a fix right line.

The most extended applicability of this case of composition of simple movements is represented by the generation by rolling of the generatrix G-involutes at the generation of the surfaces of the flanks of the cylindrical gear by different proceedings.

2.2. The composition of a rotation movement with a rectilinear translation made on a direction inclined given the rotation axe

We consider a frustum of a rotation right cone having the minimum radius r_0 in the plane YOZ of the coordinates system $OXYZ$ and with δ, the half of the angle at the peak of the précised cone. We give a rotation movement around OX axe with the frequency n (rot/min).

In the same time, an initial generating point M_0 makes a translation right linear movement on the direction of it generatrix G with the speed $\vec{v}_G$ and, after a certain time becomes in the point M generating the cinematic curve G. That curve is described by the generating point M in it resulting movement, but relative to the frustum of the rotation.

In order to obtain the relative movement, we consider the frustum of rotation fixed, by it rotation in an inverse sense with the frequency $n = -n$, on the rotation angle Φ, (Fig. 2.a).

By consequence, the speed of the relative movement of the generating point M is the vector $\vec{v}_M$, given by the vectorial relation:

$$\vec{v}_M = \vec{v}_T^{*} + \vec{v}_G. \tag{13}$$

in which $\vec{v}_T^{*}$ is the vector of the tangential speed from the rotation movement with the frequency $\left|n^{*}\right|$, corresponding to the instant speed r having the size:

$$\vec{v}_T^{*} = 2\pi rn. \tag{14}$$

The translation right line movement is a uniform movement made by the generator point M along the generatrix G of the frustum of the rotation right cone, on cyclical cinematic constant distances, having the size p_E. A cinematic cycle is formed by a rotation of the frustum of the rotation right cone, and is made in the time T_{CIC}, having the size:

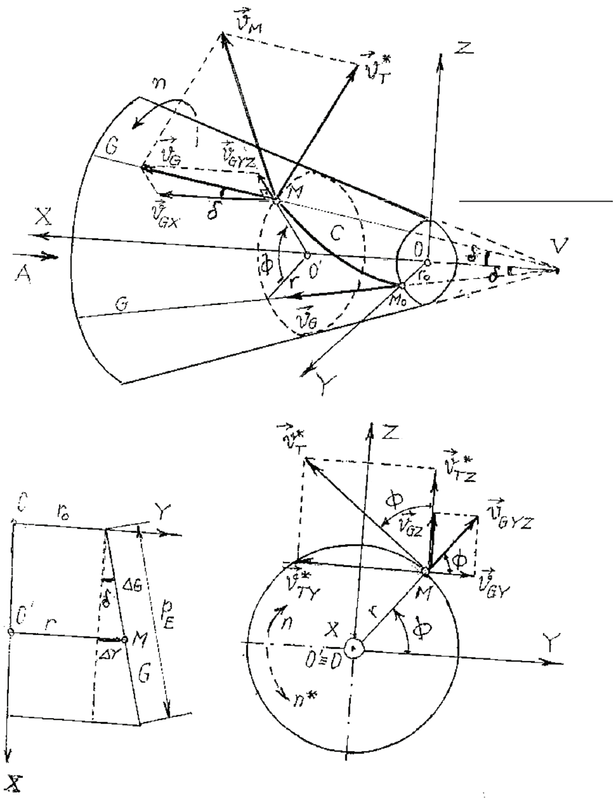

Fig. 2. The generation of a conical helix.

$$T_{CIC} = \frac{1}{n}. \tag{15}$$

In this case the speed size $\vec{v}_G$ is:

$$v_G = \frac{p_E}{T_{CIC}} = p_E n. \tag{16}$$

It results in this case:

$$R_{CCIN} = \frac{v_G}{n} = p_E. \tag{17}$$

that imposes to the two movements that are composed the cinematic co-ordination with the view of the generating point M cross cyclical, along the generatrix G constant distances p_E.

The vector of the speed $\vec{v}_G$ is projected on the axe OX and on the plane YOZ, after the vectorial relationship:

$$\vec{v}_G = \vec{v}_{GX} + \vec{v}_{GYZ}. \tag{18}$$

in which the components $\vec{v}_{GX}$ and $\vec{v}_{GYZ}$ are the sizes:

$$\begin{aligned} v_{GX} &= v_G \cos\delta = p_E n \cos\delta, \\ v_{GY} &= v_G \sin\delta = p_E n \sin\delta. \end{aligned} \tag{19}$$

At their turn, the vector $\vec{v}_{GYZ}$ is projected on the axes OY and OZ, and they are obtained the components $\vec{v}_{GY}$ respectively $\vec{v}_{GZ}$, with the sizes:

$$\begin{aligned} v_{GY} &= v_{GYZ} \cos\Phi = p_E \sin\delta\cos\Phi, \\ v_{GZ} &= v_{GYZ} \sin\Phi = p_E \sin\delta\sin\Phi. \end{aligned} \tag{20}$$

If the vectorial relation (13) is projected on the co-ordinate system axes $OXYZ$ the size of the components of the speed $\vec{v}_M$ on these axes are obtained, with the relationships:

$$\begin{aligned} v_{MX} &= v_{GX} = p_E n \cos\delta \\ v_{MY} &= v_{GY} - \left|v^*_{TY}\right| = v_{GY} - \left|v^*_T\right| \sin\Phi = \\ &= p_E n \sin\delta\cos\Phi - 2\pi r n \sin\Phi, \\ v_{MZ} &= v_{GZ} - \left|v^*_{TZ}\right| = v_{GZ} - \left|v^*_T\right| \cos\Phi = \\ &= p_E n \sin\delta\sin\Phi + 2\pi r n \cos\Phi. \end{aligned} \tag{21}$$

We can observe from the Fig. 2.b that the size of the instant radius is given by the relationship:

$$r = r_0 + \Delta Y = r_0 + \Delta G \sin\delta. \tag{22}$$

The ΔG space covered by the generating point M on the generatrix G of the frustum of the rotation right cone at a cinematic cycle is the distance p_E. So, for the rotation angle Φ the space has the size:

$$\Delta G = \frac{p_E}{2\pi}\Phi. \tag{23}$$

Replacing the relation (23) in the relation (22), we obtain:

$$r = r_0 + \frac{p_E}{2\pi}\sin\delta\cdot\Phi. \tag{24}$$

Now, we replace the relation (24) in the relations (21) and we obtain the final expressions of the of the components' size of the speed $\vec{v}_M$ on the co-ordinate axes:

$$\begin{aligned} v_{MX} &= p_E n \cos\delta, \\ v_{MY} &= p_E n \sin\delta\cos\Phi - \\ &\quad - p_E n \sin\delta\cdot\Phi\cdot\sin\Phi - 2\pi r_0 n \sin\Phi, \\ v_{MZ} &= p_E n \sin\delta\sin\Phi + \\ &\quad + p_E n \sin\delta\cdot\Phi\cdot\cos\Phi 2\pi r_0 n \cos\Phi. \end{aligned} \tag{25}$$

Similarly with the generation of the involutes, the differential co-ordinates of the generating point $M(\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z)$ are differential spaces covered respectively with the speeds v_{MX}, v_{MY} and v_{MZ} in the differential time given by the relation (8), in which:

$$\omega = \frac{1}{2\pi n} \quad [\mathrm{min}^{-1}]. \tag{26}$$

These coordinates have consequently the expressions:

$$\begin{aligned} \mathrm{d}X &= v_{MX}\cdot\mathrm{d}T, \\ \mathrm{d}Y &= v_{MY}\cdot\mathrm{d}T, \\ \mathrm{d}Z &= v_{MZ}\cdot\mathrm{d}T. \end{aligned} \tag{27}$$

We replace the relations (8), (25) and (26) in (27) and we obtain the expressions of the instant coordinates X, X, and Z of the generating point M.

$$X = \int v_{MX}\mathrm{d}T = \frac{1}{2\pi n}\int p_E n \cos\delta \mathrm{d}\Phi = \frac{p_E}{2\pi}\cos\delta\int \mathrm{d}\Phi,$$

$$\begin{aligned} Y = \int v_{MY}\cdot\mathrm{d}T &= \frac{1}{2\pi n}(p_E n \sin\delta\cos\Phi - \\ &- p_E n \sin\delta\cdot\Phi\cdot\sin\Phi - \\ &- 2\pi r_0 n \sin\Phi)\mathrm{d}\Phi = \frac{p_E}{2\pi}\sin\delta\int\cos\Phi\mathrm{d}\Phi - \end{aligned}$$

$$-\frac{p_E}{2\pi}\sin\delta\int\sin\Phi\mathrm{d}\Phi \ -r_0\int\sin\Phi\mathrm{d}\Phi,$$

$$Z=\int v_{MZ}=\frac{1}{2\pi n}(p_E n\sin\delta\sin\Phi+$$
$$-p_E n\sin\delta\cdot\Phi\cdot\cos\Phi+ \quad (28)$$
$$+2\pi r_0 n\cos\Phi)\mathrm{d}\Phi=\frac{p_E}{2\pi}\sin\delta\int\sin\Phi\mathrm{d}\Phi+$$
$$+\frac{p_E}{2\pi}\sin\delta\int\cos\Phi\mathrm{d}\Phi \ +r_0\int\cos\Phi\mathrm{d}\Phi.$$

After the effectuation of the integral,, we obtain the coordinates *X*, *Y* and *Z* of the point *M*, that represent the parametric equations of the cinematic curve *C*, described by the generating point *M* as consequence of the composition of the two movements:

$$X=\frac{p_E}{2\pi}\cos\delta\Phi+CF_X, \quad (29,a)$$

$$Y=\frac{p_E}{2\pi}\sin\delta\Phi\cos\Phi+r_0\cos\Phi+C_Y,$$
$$Z=\frac{p_E}{2\pi}\sin\delta\Phi\sin\Phi+r_0\sin\Phi+C_z. \quad (29,b)$$

in which the integration constants C_X, C_Y and C_Z can be determined knowing that at $\Phi = 0$: $X=0$, $Y=r_0$ and $Z=0$, resulting: $C_X = 0$, $C_Y = 0$ and $C_Z = 0$.

By consequence, the parametric equations of the cinematic curve *C* are:

$$X=p_0\cos\delta\cdot\Phi,$$
$$Y=p_0\sin\delta\cdot\Phi\cos\Phi+r_0\cos\Phi, \quad (29)$$
$$Z=p_0\sin\delta\cdot\Phi\sin\Phi+r_0\sin\Phi.$$

These equations represent the parametric equations of the conical helix generate on a frustum of rotation of a right cone.

In these equations, the size p_E represents the step of the conic helix measured on the generatrix *G*, and $p_0 = = p_E/2\pi$ is named the parameter of the conical helix corresponding to the generatrix *G*.

We can observe that taking $\delta = 0$ in the relations (30), the frustum of the rotation of a right cone with a minimum radius r_0 becomes the right rotation cylinder with the radius r_0, and the equations (30) becomes the parametric equations of the circular helix generated on the cylinder with the r_0 radius, having the knew expressions:

$$X=p_0\cdot\Phi,$$
$$Y=r_0\cos\Phi, \quad (31)$$
$$Z=r_0\sin\Phi.$$

In a syntheses, by the composition of a rotation movement with a right line translation movement realized in an axial plane containing the rotation axis, on an inclined direction given the axis, a generating point *M* generates as a cinematic curve *C* a conic curve, with the condition to respect R_{CCIN} given by the relationship (17), that impose to the two movements the cinematic coordination, so that the generating point M can generate the helicoidally conical surface with the given step p_E.

Practically, at the generation of the real surfaces, the cinematic curve *C* as conic helix it is generated as a directrix *D* of the surface. In this sense, an extended application of this case of composition of the simple movements is represented by the generation of the helicoidally surfaces of the conic thread.

3. CONCLUSION

The aspects made evident in the paper represent original contributions in the domain of the cinematic generation of the involutes with a fixed straight line and of the conical helix, curves employed as generatrix *G* and as directrix *D* at the generation of the complex surfaces like the involutes surfaces of the cylindrical gears and, respectively at the conic helicoidally surfaces.

All these presented aspects, presented by unique mathematical demonstrations, determine the creation and the improvement of the cinematic theory of generation of the plane and space curves by composition of simple movements.

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