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MARKOV CHAINS APPLICATIONS IN THE MANUFACTURING SCHEDULE

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Abstract: *The basic theoretical elements of the Markov chains are reviewed in the first part of the paper. An algorithm for the manufacturing schedule is developed, based on Markovian property. The manufacturing of a workpiece is analysed as numerical example. Different software for this type of problems is indicated in the last part of the paper.*

Key words: *Markov chains,transitional probabilities, manufacturing process.*

1. INTRODUCTION

The study of the sequences of independent random variables is made of classical probability theory.

A sequence of chance experiments is an independent trial process, if the possible outcomes for each experiment are the same. In the modern probability theory are considered too sequences of dependent random variables, for which the knowledge of previous outcomes influences predictions for future experiments[1].

In 1907, Andrei A. Markov introduced a new type of process, for which the outcome of a given experiment can affect the outcome of next experiment, but is conditionally independent of the past, is memory less. A process having markovian propriety is called a Markov chain. Given a finite or countable set $S = \{i_1, i_2 ...., i_n ...\}$, a sequence of random variables $X_1, X_2, ... X_n, ...$ taking values in *S* is said to posses the markovian propriety if

$$P_r(\mathrm{X}_{n+1} = i_{n+1} \mid \mathrm{X}_n = i_n, ..., \mathrm{X}_1 = i_1) = \\ = P_r(\mathrm{X}_{n+1} = i_{n+1} \mid \mathrm{X}_n = i_n)$$

The collection of random variables is called *Markov process* and the set S is the *state space* of the process. The process starts in one of states and moves successively from one state to another. The definition states that the past history is forgotten and only the current state matters in determining the future behaviour. A Markov process is said to have a stationary transition probabilities if $P_r(X_{n+1} = j \mid X_n = i)$ does not depend on *n*. In this case the process is called Markov chain and the matrix $P = p_{ij} = P_r(X_{n+1} = j \mid X_n = i)$ is named the *transition matrix*. The joint distribution of the random variables is not determined merely by the transition probabilities. It is necessary to specify the initial distribution of the starting state for example by specifying a particular state as the starting state. We refer to the row vector $\Pi = (\pi_1, \pi_2, ...)$, where $P(X_0 = i) = \pi_i$, for all $i \in S$, as the initial distribution for the Markov chain.

If *P* is a stationary transition matrix, then:

$$P_r(X_{n+1} = j \mid X_n = i) = P_{ij}^{\ n}, i, j \in S, n = 0, 1...$$

and

$$P_r(X_n = i) = (\Pi P^n)_i, i \in S, n = 0, 1...$$

In 1913, in the paper [3], Markov chose a sequence of 20 000 letters from Pushkin's Eugen Oneghin, where the two states denoted the vowels and consonants, to see if this sequence can be considered a chain. He determined the Markov chain with transition matrix:

$$\begin{array}{cc} & \text{vowel consonant} \\ \text{vowel} & \\ \text{consonant} & \end{array} \begin{bmatrix} 0.128 & 0.872 \\ 0.663 & 0.337 \end{bmatrix}.$$

In most manufacturing processes, calculation of probabilities of a system at different states, based on markovian property, plays an important part, to aid the decision making in an effective way.

A state j is said to be *accessible* from state i if, there is a positive probability of getting from state i to state j in a finite number of steps. A state i is said to communicate with state j if it is true that both i is accessible from j and j is accessible from i (written $i \leftrightarrow j$). Because "$\leftrightarrow$" defines an echivalence relation on S, the state space is partitioned into disjoint classes.

A state is said to be *recurrent* if the probability of returning to state i in finitely many steps is 1; if this probability is less then 1, "< 1", the state i is said to be *transient*. For a transient state, the expected number of returns to the state, given that is started at time 0, is finite. For the recurrent states the expected number is infinite. A state i of a Markov chain is called *absorbing* if $p_{ii} = 1$.

A Markov chain is *absorbing* if it has at least one absorbing state, and if from every state more it is possible to go to an absorbing state in one step or more steps [2]. In an absorbing Markov chain, a state which is not absorbing is called transient [5].

2. ABSORBING MARKOV CHAINS

Let an arbitrary absorbing Markov chain. Renoted the states so that the transient states come first. The transition matrix will have the the following canonical form:

$$P = \begin{bmatrix} T & Q \\ 0 & I \end{bmatrix}. \tag{1}$$

Here T is a square submatrix and represents transition matrix between transient states, the submatrix Q represents transition matrix between transient states and absorbing states and I is a square indentity matrix. In an absorbing Markov chain, the probability that the process will be absorbed is considered as:

$$T^n \to 0 \text{ as } n \to \infty . \tag{2}$$

Let j a transient state and ν_j the random variable, which represents the number of appearances of j. It defines the random variable u_j^k, $k \geq 0$, which counters the appearances of state j as:

$$u_j^k = \begin{cases} 1 \text{ if } & X_k = j \\ 0 \text{ if } & X_k \neq j \end{cases}. \tag{3}$$

Obviously $\nu_j = \sum_{k\geq 0} u_j^k$ and follows:

$$M_i(\nu_j) = \sum_{k\geq 0} 1 * P_i(X_k = j) + 0 * P_i(X_k \neq j) = \sum_{k\geq 0} p_{ij}^{(k)}, \tag{4}$$

where P_i represents the probability in the case in which the initial distribution is *concentrated* in the state i, consequently in the case in which the chain starts from the state i, while M_i represents the mean value, calculated relatively to probability P_i. The relationship (3) should be matricialy written:

$$(M_i(\nu_j))_{i,\,j\in T} = \sum_{k\geq 0} T^k = (I-T)^{-1} = I + T + T^2 + \ldots \tag{5}$$

The matrix:

$$N = (I-T)^{-1} \tag{6}$$

is called *the fundamental matrix* of the Markov chain. The *ij*-entry n_{ij} of the matrix N is the espected number of times the chain is in state j, given that it starts in state i. With the help of fundamental matrix it should be calculated the second order moment and the variance of the random variable ν_j. It can be proved that the dimensions $M_i(\nu^2_j)$ are finite and:

$$N_2 = (M_i(\nu_j^2))_{i,\,j\in T} = N(2N_{dg} - I), \tag{7}$$

where N_{dg} represents the matrix which it obtains from N equaling with zero the elements from outside the principal diagonal.

It results that the variances are given by the matrix:

$$D^2 = (M_i(\nu_j^2) - M_i^2(\nu_j))_{i,\,j\in T}. \tag{8}$$

Also with the help of the fundamental matrix it should be calculated the probabilities, notated f_{ij}, so that the chain arrives in the state j, departing from the transient state i:

$$F = (f_{ij})_{i,\,j\in T} = (N-I)N_{dg}^{-1}. \tag{9}$$

Let ν the random variable which represents the time which the chain spends within the set T, namely:

$$\nu = \sum_{j\in T} \nu_j . \tag{10}$$

The ν variable is called *absorbing time*. The vector of mean values of the absorbing time, noted:

$$m = (m_i)_{i\in T} \tag{11}$$

is given by:

$$m = (M_i(\nu))_{i\in T} = N \cdot \mathbf{e}, \tag{12}$$

where $\mathbf{e}$ is the unity vector.

Let a_{ik} be the probability that an absorbing chain will be absorbed in the absorbing state k, if it starts in the transient state i. Let A be the matrix with entries a_{ik}; then:

$$A = (a_{ik})_{i\in T, k\in S\setminus T} = NQ. \tag{13}$$

As example lets be an absorbing chain, , having the absorbing states l + 1 and l + 2 and the transient states 1,…, l. The canonic form of its transition matrix is:

$$P = \begin{array}{c} \\ l+1 \\ l+2 \\ 1 \\ 2 \\ \vdots \\ l \end{array} \begin{array}{c} \text{l+1 l+2 1 2 . . . l} \\ \begin{pmatrix} 1 & 0 & 0 & 0. & .. & 0 \\ 0 & 1 & 0 & 0. & .. & 0 \\ 0 & q_1 & r_1 & p_1 & \ldots & 0 \\ 0 & q_2 & 0 & r_2 & \ldots & 0 \\ \vdots & & & & & \\ p_l & q_l & 0 & 0 & .. & r_l \end{pmatrix} \end{array}. \tag{14}$$

Using the relationships (6)-(13) it should be obtained for every $i, j \in \overline{1, l}$:

$$n_{ij} = M_i(\nu_j) = \begin{cases} \dfrac{p_i \ldots p_{j-1}}{(1-r_i)\ldots(1-r_j)}, & i < j, \\ \dfrac{1}{(1-r_i)}, & i = j, \\ 0, & i > j. \end{cases} \tag{15}$$

The elements n_{ij} of the fundamental matrix represents the mean time in which the workpiece is in the manufacturing stage j, prior to becoming end product, or scrap, if the workpiece starts its manufacturing cycle in the stage i.

$$M_i(\nu_j{}^2) = \begin{cases} \dfrac{p_i \ldots p_{j-1}}{(1-r_i)\ldots(1-r_j)} \cdot \dfrac{1+r_j}{1-r_j}, & i < j, \\ \dfrac{1+r_i}{(1-r_i)^2}, & i = j, \\ 0, & i > j, \end{cases} \tag{16}$$

$$f_{ij} = \begin{cases} \dfrac{p_i \cdots p_{j-1}}{(1-r_i)\dots(1-r_{j-1})}, & i < j, \\ r_i, & i = j, \\ 0, & i > j, \end{cases} \tag{17}$$

and for every $i \in \overline{1, l}$:

$$m_i = M_i(\nu_j) = \sum_{j=1}^{l} n_{ij} =$$

$$= \begin{cases} \dfrac{1}{1-r_i} + \sum\limits_{j=1}^{l} \dfrac{p_i \cdots p_{j-1}}{(1-r_i)\dots(1-r_j)}, & 1 \le i < l, \\ \dfrac{1}{(1-r_i)}, & i = l, \end{cases} \tag{18}$$

$$\begin{cases} a_{i,l+1} = \dfrac{p_i \cdots p_l}{(1-r_i)\dots(1-r_l)}, \\ a_{i,l+2} = 1 - a_{i,l+1}. \end{cases} \tag{19}$$

Assuming that the workpiece taked into account is a shaft which is manufactured through a technological process composed from four stages: first – cutting, then welding, heat treatment (annealing) and then the final quality control, evolving in automated cycle, we obtained the following results from experimental data:

$$P = \begin{pmatrix} 0.02 & 0.88 & 0 & 0 & 0 & 0.10 \\ 0 & 0.04 & 0.90 & 0 & 0 & 0.06 \\ 0 & 0 & 0.02 & 0.94 & 0 & 0.04 \\ 0 & 0 & 0 & 0.01 & 0.97 & 0.02 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

Applying the relations (14-19) it obtains the subsequent results:

$$N_2 = \begin{pmatrix} 1.0204 & 0.9353 & 0.8590 & 0.8156 \\ 0 & 1.0416 & 0.9566 & 0.9083 \\ 0 & 0 & 1.0204 & 0.9688 \\ 0 & 0 & 0 & 1.0101 \end{pmatrix} \tag{21}$$

The elements under the principal diagonal are zero because the workpiece can not pass in an subordinate manufacturing stage reportedly to the state where it is. It can be observed that the values on the lines are decreasing, from the left to the right, so that, as the workpiece advances in the technological process, these mean times become smaller.

$$N_2 = \begin{pmatrix} 1.062 & 1.0133 & 0.8940 & 0.8321 \\ 0 & 1.1284 & 0.9956 & 0.9266 \\ 0 & 0 & 1.0620 & 0.9884 \\ 0 & 0 & 0 & 1.0305 \end{pmatrix}, \tag{22}$$

$$D^2 = \begin{bmatrix} 0.0208 & 0.1384 & 0.1561 & 0.1669 \\ 0 & 0.0434 & 0.0805 & 0.1011 \\ 0 & 0 & 0.208 & 0.0497 \\ 0 & 0 & 0 & 0.0102 \end{bmatrix}, \tag{23}$$

$$F = \begin{bmatrix} 0.02 & 0.8980 & 0.8418 & 0.8075 \\ 0 & 0.04 & 0.9375 & 0.8992 \\ 0 & 0 & 0.02 & 0.9592 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}. \tag{24}$$

The elements f_{ij} of the matrix F represent the probability that the workpiece found into the manufacturing stage i will arrive in the stage j. It can be observed that $f_{ii} << f_{ij}, i < j,$ since is more credible that the workpiece pass into a superior stage than to be reworked

$$m = \begin{bmatrix} 3.603 \\ 2.9066 \\ 1.9892 \\ 1.0101 \end{bmatrix}. \tag{25}$$

The components of the vector m represent the mean manufacturing working time until it obtains the finished product, respectively scrap, if the workpiece is initialy in the i manufacturing stage. As sun as the workpiece is into a more advanced manufacturing stage, the more so as its mean duration is smaller.

In the analised case, by example, the mean manufacturing duration, when the workpiece is initialy at the first stage, is 3.6302 time units, while it is at the thirth stage, the mean manufacturing duration will be 1.9892 time units

$$\mathrm{A} = \begin{bmatrix} 0.7911 & 0.2089 \\ 0.8811 & 0.1189 \\ 0.9398 & 0.0602 \\ 0.9798 & 0.0202 \end{bmatrix}. \tag{26}$$

The dimensions $a_{i,l+1}$, $a_{i,l+2}$ represent the probabilities that the workpiece beeing into the i stage to become finished product, respectively scrap.

As consequence, for the data from above, $a_{15} = 0.7911$, that it can be translated as the probability that the workpiece found into the first manufacturing stage becomes finished product, namely generalizing, a_{i5} is the reliability of the workpiece technology, which was initialy found at the i stage, but $a_{16} = 0.2089$ is the probability to become scrap.

The model presented above permits to calculate the mean losses, knowing for every manufacturing stage the probabilities that the workpiece to become scrap and the prices of the scrap workpiece after every manufacturing stage.

3. GRAPHS AND APPLICATIONS

Let X_n a finite Markov, homogeneous, with the state set $S = \{1, 2 \dots n\}$, Π_0 - the initial distribution and $P = (p_{ij})$ the transition matrix. The graph, noted $G = (S, \Gamma)$, where Γ is a subset of the cartesian product S^2, definited through the relationship:

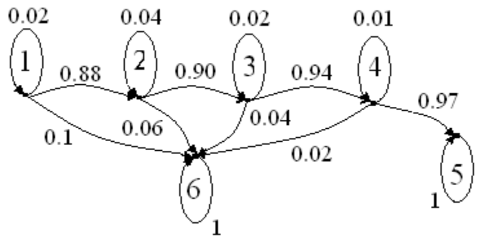


Fig. 1. The graph associated to the Markov chain.

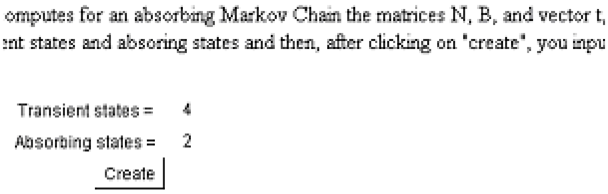
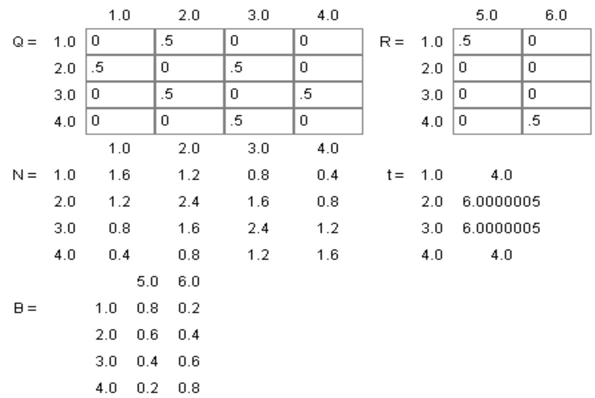


Fig. 2. The Java applet for Markov chain.



Q =

	1.0	2.0	3.0	4.0
1.0	0	.5	0	0
2.0	.5	0	.5	0
3.0	0	.5	0	.5
4.0	0	0	.5	0

R =

	5.0	6.0
1.0	.5	0
2.0	0	0
3.0	0	0
4.0	0	.5

N =

	1.0	2.0	3.0	4.0
1.0	1.6	1.2	0.8	0.4
2.0	1.2	2.4	1.6	0.8
3.0	0.8	1.6	2.4	1.2
4.0	0.4	0.8	1.2	1.6

t =

1.0	4.0
2.0	6.0000005
3.0	6.0000005
4.0	4.0

B =

	5.0	6.0
1.0	0.8	0.2
2.0	0.6	0.4
3.0	0.4	0.6
4.0	0.2	0.8

Fig. 3. Results obtained with Java applet for Markov chain.

$$(i, j) \in \Gamma \Leftrightarrow p_{ij} > 0,$$

is named the *graph* associated to the Markov chain. Markov chains can be described by a graph. The edged are labeled by the transitional probabilities of going from one state to the other states. In our case the graph is presented in Fig. 1.

In a complex technological process the calculus becomes laborious [4] and it is useful the utilization of a adequate software: basically, having the initial distribution and the matrices of the transition probabilities, even in the stationary case, then in the unstationary, general one, the problem is effectively reduced to the matrix calculus. As soon as it is not available a professional software for mathematics /statistics like Mathcad, Mathlab etc. it is always possible to use with more effort very popular programs as Excel, or is possible to work with some freeware programs as XNUMBERS MULTI - PRECISION FLOATING POINT COMPUTATION and NUMERICAL METHODS for EXCEL-XNUMBERS.XLA - Ver. 5.5 - Update of Aug. 2007 [6], very easy to install and working under Excel. The large interest for Markov chains is illustrated by the development of a special software like the Program Absorbing Chain, which calculates the basic descriptive quantities of an absorbing Markov chain, from the Dartmouth University [7]. The software can be used on line, with Java Applets (Figs. 2 and 3) (the notations have a slightly different meaning compared with the above calculus).

4. CONCLUSIONS

The theoretical bases of the presented model are approached in the related literature (Iosifescu, Karlin). The main part of the paper is the application of the mathematical theory at the usual manufacturing problems. The study utility consists in the fact that the obtained results can be used for an effective design of the actual strategies in manufacturing.

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